SOLUTIONS.

2894 [1921, 184]. Proposed by PHILIP FRANKLIN, Harvard University, AND E. L. POST, Columbia University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates*.

III. For every m-class there is at least one *conjugate m-class*.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry*, vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

SOLUTION BY THE PROPOSERS WITH IMPROVEMENTS SUGGESTED BY W. E. CLELAND, R. HARTSHORNE, and G. E. RAYNOR, Princeton University.

Examples to prove the independence of assumptions I to VII.

I. The elements $A\ B\ C\ D$ and the sets AB and CD.

II. The combinations of six letters taken three at a time.

III. The elements $A\ B$ and the set AB.

IV. The elements $A\ B\ C$, $X\ Y\ Z$ and the sets: $ABZ, ACY, BCX, AX, BY, CZ, XY, YZ, XZ$.

V. S is the null class, and there are no m-classes.

VI. S is the null class, and there are two null-m-classes.

VII. S contains an infinite number of elements, and there are an infinite number of m-classes formed as follows: Start with two m-classes $A_1 A_2$ and $A_3 A_4$. Form all the classes required by I. Form new classes $B_1 B_2$, $B_3 B_4$, which are to be conjugate to these last. Satisfy IV by inserting in the classes common elements, $C_1, C_2, \cdots$, never using the same element in more than one pair of classes. Form all the classes with these new elements required by I, and repeat the process.

THEOREM 1. *Every m-class contains at least two elements.*

For suppose one m-class contained a single element A. By III there would exist an m-class not containing A, and by VI it would contain an element B. By I there would be a class AB and by III, a conjugate class containing neither A nor B. Thus we would have two m-classes conjugate to the class with a single element A, contradicting IV.

THEOREM 2. *S contains at least four elements.*

By V there exists at least one m-class, which by theorem 1 contains at least two elements. By III there exists a conjugate m-class which also contains at least two elements.

THEOREM 3. *S contains at least six m-classes.*

The previous proof shows the existence of two conjugate m-classes, each containing at least two elements. Each of the elements of one of these m-classes, with one of the elements of the second of these classes, by I determines an m-class, and these m-classes, at least four in number, are all distinct, since if two coincided, by II it would coincide with one of our original classes, and these would not be conjugate.

THEOREM 4. *No m-class contains more than two elements.*

Let A and C be any two non-conjugate m-classes. By VII we may suppose that the number of elements in A, say p, is equal to or greater than the number of elements in C. Let the elements of A be $A_1, A_2, \cdots, A_p$, A_1 the element common to A and C.

Let $B_1, B_2, \cdots$, be the elements of B, the m-class conjugate to A, B_1 the element common to B and C. C then contains A_1 and B_1, and if there are any other elements in C we may call them $C_1, C_2, \cdots, C_k$, so that $k \leqq p - 2$.

Now if B_x is any one of the B's except B_1, the m-classes $B_xA_2, \cdots, B_xA_p$ will be distinct and no one of them can contain A_1 or B_1. Otherwise one of them would be the class A and contain B_x, or the class B and contain one of the A's, which is impossible, since A and B are conjugate

classes. One of them may be the conjugate of C, but each of the others, if $p > 2$, must contain one of the elements of C by IV, and that must be one of the C's and not A_1 or B_1. On the other hand, no two of them can contain the same C for they already have B_x in common. Therefore, there must be at least $p - 2$ of the C's, and, as k is not greater than $p - 2$, it follows that $k = p - 2$ and that C contains the same number of elements as A.

It follows also that one of these classes is the conjugate of C; that is, that the conjugate of C contains B_x, which is any one of the B's except B_1. This means that there can be only one B besides B_1, and that the class B can contain only two elements.

Now C was any class non-conjugate to A. Therefore, every class except B contains p elements while B contains only two. But if we start with some other class in place of A, we can prove that its conjugate contains two elements and that B contains p. This is possible only if $p = 2$ and every class contains only two elements.

THEOREM 5. *The elements of S and the m-classes are isomorphic with the elements A, B, C, D and the sets AB, AC, AD, BC, BD, CD.*

The four elements of theorem 2 may be labelled A, B, C, D and the six m-classes of theorem 3 the six of this proposition. By theorem 4 there are no more elements in any of these m-classes, and by II no more m-classes containing only these elements. Let X be an additional element. By I there is a class AX, and by theorem 4, it contains no more elements. We then have classes AX and AB both conjugate to CD, and since this violates IV it proves that there are no elements X. The six sets above evidently satisfy I to VII.

NOTE. In connection with the above solution Professor Veblen's comments on the origin and discussion of the problem will be of interest to the readers of the MONTHLY.

Professor VEBLEN says, "The problem originated in my course in Projective Geometry which I began, as usual, by a discussion of the abstract point of view in mathematics. In order to emphasize the point that our logical processes should be independent of any particular set of mental images or, indeed, of any knowledge of what the propositions are about, I proposed that certain members of the class should make up a set of postulates which would give the properties of some set of objects chosen but not divulged by them. The other students were then challenged to make logical deductions from the postulates and thereby deduce enough theorems to learn what the postulate makers had in mind.

"As a result of this suggestion, Mr. Post and Mr. Franklin brought forward a set of postulates which is essentially the one offered in the enclosed problem and a solution was found by several of the other students in precisely the manner indicated. One of the students indeed went further and pointed out that one of the postulates in the original set was redundant.

"The exercise was a success in showing how mathematical deductions can be made without knowledge as to what one is reasoning about. It also brought out vividly the problem of the significance of the logical processes as a method of discovery. While there is a sense in which it is true that you cannot get anything out of a set of postulates except what has been put in them, you can at least find out what was put in them. This is what the solver of this problem has to do in a simple case. In the more complicated case of ordinary geometry the student is apt to think that he understands what is in the axioms, but every time that he witnesses the derivation of a new theorem it turns out that there was something in them that he had not seen before.

"By a slight modification this problem can also be used to propose another problem which it seems to me may turn out to be an important one. Let us replace Assumption VII by the assumption that no m-class contains more than 6 elements. The mathematical science based on the assumptions could then be built up without ever counting beyond, let us say, 24. The question arises, how much of logic is needed to develop so limited a mathematical science? Would it be possible to single out a subset of the postulates of logic which would suffice for the purpose? If so, what processes of logic can be omitted? What sort of a logic results if the omitted processes are replaced by others? If it should turn out that the logic required for a satisfying theory of this finite system (or any other partciular system as, for example, a particular finite projective space) stops short of that required for larger systems, we would be in the presence of a criterion for the classification of logical processes which might help toward deciding the question as to what logical processes are legitimate in dealing with various types of infinite sets."

2899 [1921, 228]. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.